Pre-Calculus Notes

Name: Key

Section 9.3 – Geometric Sequences and Series

GEOMETRIC SEQUENCE: a sequence where each term is obtained by multiplying the previous term by a given number (common ratio)

Common Ratio: the quotient of any term by previous term $\rightarrow \frac{a_{n+1}}{a_n}$

Today we are going to look for a "simple" way to find an explicit formula for a geometric sequence. Consider the following sequence. Do you see a pattern for obtaining each term?

a_1	a_2	a_3	a_4	a_5
5	15	45	135	405
5	(5X3)	(5X3)X3	((5X3)X3)X3	(((5X3)X3)X3)X3
$5(3)^0$	$5(3)^1$	$5(3)^2$	$5(3)^3$	$5(3)^4$

Therefore the ***EXPLICIT FORMULA*** is: $a_n = 5 \cdot 3^{n-1}$

What is the ***RECURSIVE FORMULA?*** $a_1 = 5 \quad a_{n+1} = 3 \cdot a_n$

General Form of a Geometric Sequence:

Recursive: $a_1,\ a_{n+1} = a_n(r)$ **Explicit:** $a_n = a_1 \cdot r^{n-1}$

Example 1: For the following geometric sequences, find both the explicit and recursive formulas.

	Sequence	Recursive Formula AND Explicit Formula
a)	2, -12, 72, -432, ... −6 −6 −6	$a_1 = 2 \quad a_{n+1} = -6a_n$ $a_n = 2 \cdot (-6)^{n-1}$
b)	9, 7.2, 5.76, 4.608, ... ×0.8 ×0.8 ×0.8	$a_1 = 9, \quad a_{n+1} = 0.8a_n$ $a_n = 9(0.8)^{n-1}$

c) $a_1 = 5$, $r = \frac{3}{2}$ First 5 terms are: 5 $\frac{15}{2}$ $\frac{45}{4}$ $\frac{135}{8}$ $\frac{405}{16}$	$a_1 = 5,\ a_{n+1} = \frac{3}{2}a_n$ $a_n = 5 \cdot \left(\frac{3}{2}\right)^{n-1}$

How could we use our calculator and table to find these values AND keep the answers in fraction format?

$y_1 = 5 \cdot 3^{(n-1)}$ $\quad y_2 = 2^{(n-1)}$

Example 2: Find the explicit formula and the indicated term.

a) $a_4 = 64$, $r = \frac{1}{4}$ $\quad a_n = 4096\left(\frac{1}{4}\right)^{n-1}$ $\quad a_8 = 4096\left(\frac{1}{4}\right)^{8-1}$

$\boxed{a_8 = \frac{1}{4}}$

$a_4 = a_1 \cdot r^{n-1}$

$64 = a_1 \cdot \left(\frac{1}{4}\right)^{4-1}$

$64 = \frac{1}{64}a_1 \rightarrow a_1 = 4096$

b) 5, 30, 180... $\quad a_n = 5 \cdot 6^{n-1}$ $\quad a_{10} = 5 \cdot 6^{10-1}$

×6 ×6

$\boxed{a_{10} = 50388480}$

c) $a_2 = 3$, $a_5 = \frac{3}{64}$ $\quad a_n = 12 \cdot \left(\frac{1}{4}\right)^{n-1}$ $\quad a_1 = 12$

Could you use "table" on your calculator to check your formula and the 1st term? yes

$a_5 = a_2 \cdot r^{5-2}$

$\frac{3}{64} = 3r^3 \rightarrow \frac{1}{64} = r^3$

$r = \frac{1}{4}$

$a_2 = a_1 \cdot r^{2-1}$

$3 = a_1 \cdot \left(\frac{1}{4}\right)^1$

$a_1 = 12$

d) $a_3 = \frac{16}{3}$, $a_5 = \frac{64}{27}$ $\quad a_n = 12 \cdot \left(\frac{2}{3}\right)^{n-1}$ $\quad a_7 = \frac{256}{243}$

OR $a_n = 12\left(-\frac{2}{3}\right)^{n-1}$

Could you find the 7th term without finding the general term formula? yes

$a_7 = 12\left(\frac{2}{3}\right)^6$

$a_5 = a_3 \cdot r^{5-3}$

$\frac{64}{27} = \frac{16}{3} \cdot r^2$

$\frac{4}{9} = r^2 \rightarrow r = \pm\frac{2}{3}$

$a_3 = a_1 \cdot r^{3-1}$

$\frac{16}{3} = a_1 \cdot \left(\frac{2}{3}\right)^2$

$\frac{16}{3} = a_1 \cdot \frac{4}{9}$

$a_1 = 12$

Example 1: Find the indicated form and find the partial sum for the series.

a) $\sum_{n=1}^{5} 5\cdot 2^{n-1}$

$5 + 10 + 20 + 40 + 80$

$= 155$

b) $1 - 0.8 + 0.64 - 0.512 + \ldots$

$\sum_{k=1}^{\infty} 1\cdot(-.8)^{k-1}$

$1 - 0.8 + .64 - .512 + .4096 - .32768 + .262144$

$S_7 = 0.672064$

Just finding the terms and adding them up is good for series with a small number of terms. This is not a good method, however, if we have a large number of terms.. And your teacher may ask you to find the sum of 100 terms... what a meanie!

We need a formula!

GEOMETIC SERIES: $S_n = \dfrac{a_1(1-r^n)}{1-r}$ for $r \neq 1$

Example 2: Find each indicated partial sum using the formula.

a) $\sum_{a=1}^{\infty} \dfrac{3}{4^a}$

$\frac{3}{4}, \frac{3}{16}, \frac{3}{64}, \ldots$

$S_6 = \dfrac{\frac{3}{4}\left(1-\left(\frac{1}{4}\right)^6\right)}{\left(1-\frac{1}{4}\right)}$

$S_6 \approx 0.999756$

b) $15 - 3 + \frac{3}{5} - \ldots - \frac{3}{625}$

$= \sum_{k=1}^{6} 15\left(-\frac{1}{5}\right)^{k-1}$

$\frac{-3}{625} = 15\left(-\frac{1}{5}\right)^{k-1}$

$-\frac{1}{3125} = \left(-\frac{1}{5}\right)^{k-1}$

$\left(-\frac{1}{5}\right)^5 = \left(-\frac{1}{5}\right)^{k-1}$

$k = 6$

$S_6 = \dfrac{15\left(1-\left(-\frac{1}{5}\right)^6\right)}{\left(1-\left(-\frac{1}{5}\right)\right)}$

$S_6 = 12.4992$

Now lets look at finding the sum of a geometric series with an infinite number of terms.

Ex. 3) Find S_∞ for $\sum_{k=1}^{\infty} 3(2)^{k-1}$. $S_\infty =$ DNE (used table)

(Put the term formula in y_1 and the sum formula in y_2. Go to the table and see what happens to the terms, and the sum of the terms, as k gets larger.)

Can we find this sum? Why?

no b/c the terms and sums both approach infinity!

Ex. 4) Find S_∞ for $\sum_{k=1}^{\infty} 16\left(\frac{1}{2}\right)^{k-1}$. $S_\infty =$ 32

(Put the term formula in y_1 and the sum formula in y_2. Go to the table and see what happens to the terms, and the sum of the terms, as k gets larger.)

Can we find this sum? Why?

yes! The terms approach 0, so the sum approaches a finite value.

Why are we able to find the sum with one series and not the other?

$|r|<1$ then the terms go to zero

Now, lets look at the general formula for finding the sum of a series and ask -

What would the formula become if r is a number between -1 and 1 and n is getting very large - approaching infinity?

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r} = \frac{a_1(1-(\vec{0}))}{1-r} = \frac{a_1}{1-r}$$

Therefore, the Sum of an Infinite Geometric Series for $|r|<1$ is... $S = \frac{a_1}{(1-r)}$.

Find the sums of the following infinite geometric series.

Ex. 5) $\sum_{n=1}^{\infty} 4(.06)^{(n-1)}$ S = $\frac{4}{1-0.6} = \frac{200}{47}$

Ex. 6) 3 + 0.3 + 0.03 + 0.003 + ... S = $\frac{3}{1-.1} = \frac{10}{3}$