Pre-Calculus Notes

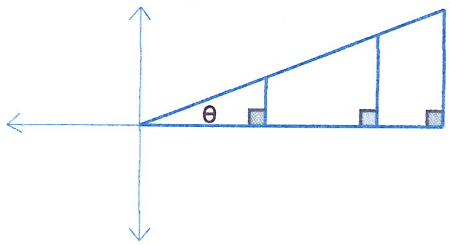
Name: Key

Sections 4.2 and 4.4 Meshed Together...

PART ONE: We looked at trigonometric functions for angles measuring between 0° and 90° when we focused on the acute angles inside of a right triangle. We can also assign trig. values to ANY angle measure, including angles greater than 90°. But need to understand **reference angles** and **reference triangles** first.

First, let's review an important geometry concept...

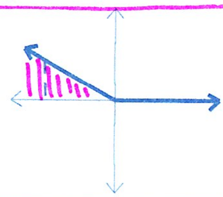
Will the angle θ have the same trig values REGARDLESS of the triangle used? yes **Why?**

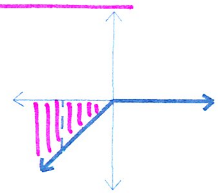


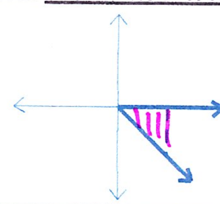
- $\sin\theta$ is a ratio
- the Δs are similar
- so their corresponding sides are proportional, so $\therefore$ same ratio!

Second, let's look at reference angles and reference triangles.

A ***reference angle*** is the ACUTE angle (always positive) formed by the TERMINAL side of any angle in standard position and the nearest portion of the x-axis. The triangle formed is the ***reference triangle***.

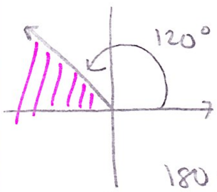






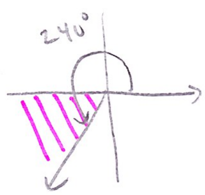
Example 1: Sketch the angle in standard position. Then shade in the reference angle and give its measure in degrees.

a. 120°



$180 - 120 = \boxed{60^\circ}$

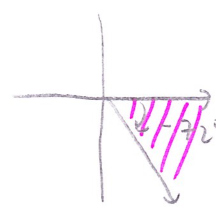
b. 240°



$240 - 180 = \boxed{60^\circ}$

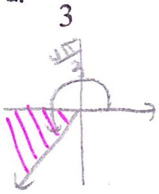
c. -72°

$\boxed{72^\circ}$



Example 2: Sketch the angle in standard position. Then shade in the reference angle and give its measure in radians.

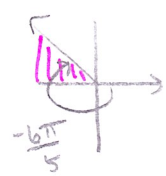
a. $\frac{4\pi}{3}$



$\frac{4\pi}{3} - \pi$

$\boxed{\frac{\pi}{3}}$

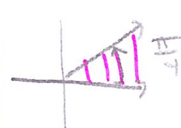
b. $-\frac{6\pi}{5}$



$\frac{6\pi}{5} - \pi$

$\boxed{\frac{\pi}{5}}$

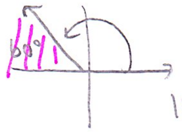
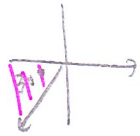
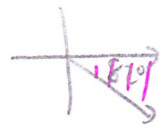
c. $\frac{\pi}{4}$



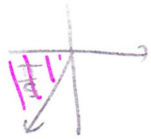
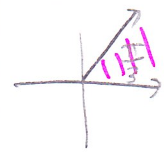
$\boxed{\frac{\pi}{4}}$

Let's change the approach. I will give you the reference angle and the quadrant that the original angle terminates in… you give the measure of an angle that has the given reference angle.

Example 3: Give the measure of an angle in degrees that has the given reference angle and that terminates in the designated quadrant.

a. 60°; terminates in Q II	b. 30°; terminates in Q III	c. 82°; terminates in Q IV
 $180-60=\boxed{120^\circ}$	 $180+30=\boxed{210^\circ}$	 $=360-82=\boxed{278^\circ}$

Example 4: Give the measure of an angle in radians that has the given reference angle and that terminates in the designated quadrant.

a. $\frac{\pi}{6}$; terminates in Q II	b. $\frac{\pi}{5}$; terminates in Q III	c. $\frac{\pi}{3}$; terminates in Q I
 $\pi-\frac{\pi}{6}=\boxed{\frac{5\pi}{6}}$	 $\pi+\frac{\pi}{5}=\boxed{\frac{6\pi}{5}}$	 $\boxed{\frac{\pi}{3}}$

PART TWO:

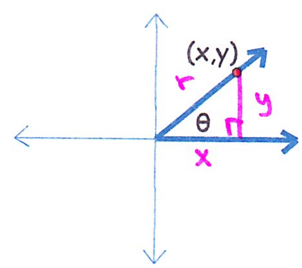
Let's start associating the trig values with x, y, and r, in addition to adj, opp, and hyp.

Let θ be an angle in standard position with (x,y) a point on the terminal side of θ and $r=\sqrt{x^2+y^2}\neq 0$, since $x^2+y^2=r^2$. Then…

$$\sin\theta=\frac{y}{r} \qquad \csc\theta=\frac{1}{\sin\theta}=\frac{r}{y}$$

$$\cos\theta=\frac{x}{r} \quad \text{and} \quad \sec\theta=\frac{1}{\cos\theta}=\frac{r}{x}$$

$$\tan\theta=\frac{y}{x} \qquad \cot\theta=\frac{1}{\tan\theta}=\frac{x}{y}$$

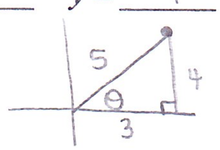


$x^2+y^2=r^2$

r is ALWAYS positive

Example 1: Find the 6 trigonometric function values of the angle with a point $(3,4)$ on the terminal side of the angle, θ.

$x=$ 3 $\quad y=$ 4 $\quad r=$ 5

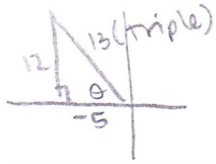


$\sin\theta=\frac{4}{5}$ $\quad \cos\theta=\frac{3}{5}$ $\quad \tan\theta=\frac{4}{3}$

$\csc\theta=\frac{5}{4}$ $\quad \sec\theta=\frac{5}{3}$ $\quad \cot\theta=\frac{3}{4}$

Example 2: Find the 6 trigonometric function values of the angle with a point $(-5,12)$ on the terminal side of the angle, θ.

$x =$ <u>−5</u> $y =$ <u>12</u> $r =$ <u>13</u>



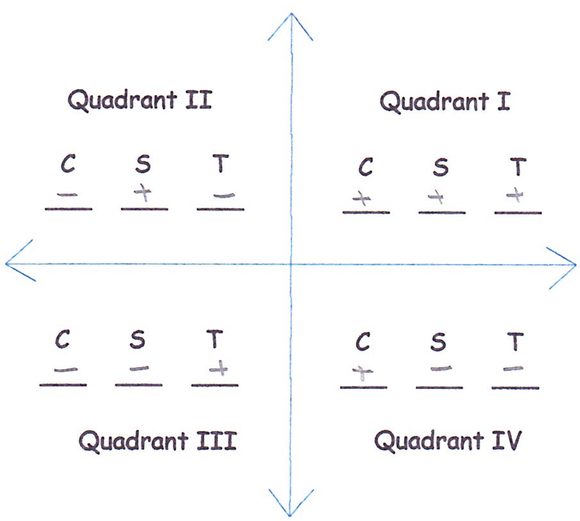
$\sin\theta = \frac{12}{13}$	$\cos\theta = \frac{-5}{13}$	$\tan\theta = \frac{12}{-5}$
$\csc\theta = \frac{13}{12}$	$\sec\theta = \frac{-13}{5}$	$\cot\theta = \frac{-5}{12}$

Now let's look at determining the sign (positive or negative) of a function by looking at the quadrant in which the angle terminates.

The signs of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because $\cos\theta = \frac{x}{r}$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.)

Where will $\sin\theta$ be positive? <u>wherever y is positive</u> → Q I and II

Where will $\tan\theta$ be positive? <u>wherever x & y are both + or −</u> → Q I and III



So… what is a trick to help me remember?

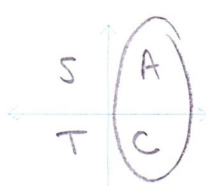
"All Stupid Teachers Cheat"
can help you identify which of the main trig. functions (sin, cos, and tan) are positive for each quadrant. Go counter-clockwise and then just remember that the trig. functions reciprocal will have the same sign (so csc is positive in all the quadrants where sin is positive).

Example 3: State the quadrant(s) in which θ terminates based on the given information.

a. $\cos\theta > 0$

positive

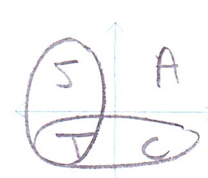
Q I and IV



b. $\sin\theta < 0$, $\cos\theta < 0$

neg neg

Q III

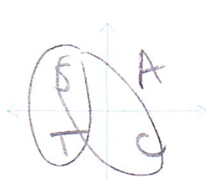


c. $\sec\theta < 0$, $\tan\theta < 0$

$\cos\theta < 0$ negative

negative

Q II

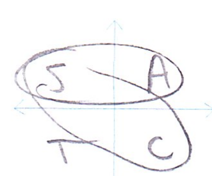


d. $\csc\theta > 0$, $\cot\theta < 0$

$\sin\theta > 0$ pos.

$\tan\theta < 0$ neg.

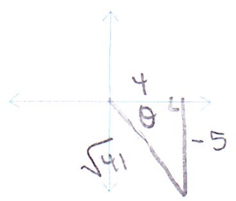
Q II



Example 4: Given $\tan\theta = -\frac{5}{4}$ and $\cos\theta > 0$ (pos), find the exact values of the other 5 functions.

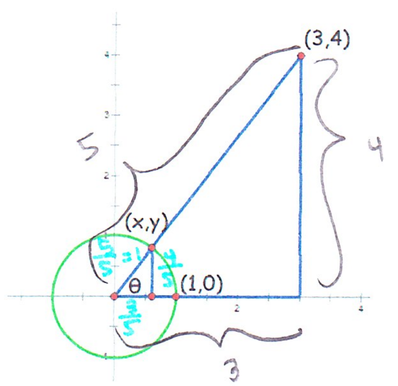
$x =$ 4 $\quad y =$ −5 $\quad r =$ $\sqrt{41}$

QIV!



$\sin\theta = \frac{-5\sqrt{41}}{41}$	$\cos\theta = +\frac{4\sqrt{41}}{41}$	$\tan\theta = -\frac{5}{4}$
$\csc\theta = \frac{\sqrt{41}}{-5}$	$\sec\theta = \frac{\sqrt{41}}{4}$	$\cot\theta = -\frac{4}{5}$

We will be focusing on the Unit Circle more and more over the next few days. What is a Unit Circle? Well, it is a circle that is centered at the origin and that has a radius of one unit. **What effect will that have?**



What are the coordinates for x and y? $\left(\frac{3}{5}, \frac{4}{5}\right)$

What is the cosine of θ? $\frac{3}{5}$ wow!

What is the sine of θ? $\frac{4}{5}$ wow!

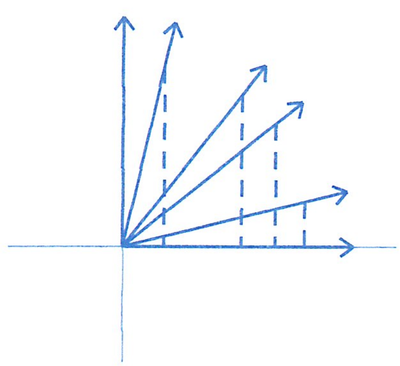
Hence, in the UNIT CIRCLE…

- **the cosine of an angle is always** its x-coordinate
- **AND the sine of an angle is always** its y-coordinate

Quadrantal Angles –

Now, we will focus on angles in standard position that terminate on an axis, also known as **quadrantal angles.**

What happens as the terminal side of the angle approaches the x-axis?

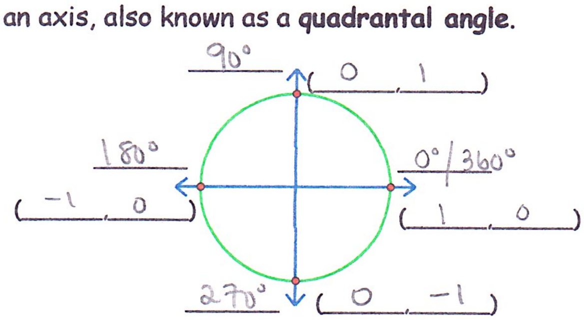


What happens when the terminal side is ON the x-axis?

What happens as the terminal side of the angle approaches the y-axis?

What happens when the terminal side is ON the y-axis?

Armed with this knowledge, let's figure out the trig function values for any angle whose terminal side lies on an axis, also known as a **quadrantal angle.**



REMEMBER…

$\sin\theta = \frac{y}{r}$		$\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$	and	$\sec\theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$		$\cot\theta = \frac{x}{y}$

Example 5: Evaluate the value of each of the six trigonometric functions for each quadrantal angle θ.

a. $\theta = 0°$ (1, 0) (cos, sin) $\tan = \frac{\sin}{\cos}$ $\sin\theta =$ 0 $\cos\theta =$ 1 $\tan\theta =$ 0 $\csc\theta =$ undef $\sec\theta =$ 1 $\cot\theta =$ undef	b. $\theta = 90°$ (0, 1) (cos, sin) $\tan = \frac{\sin}{\cos}$ $\sin\theta =$ 1 $\cos\theta =$ 0 $\tan\theta =$ undef $\csc\theta =$ 1 $\sec\theta =$ undef $\cot\theta =$ 0
c. $\theta = -180°$ (-1, 0) $\sin\theta =$ 0 $\cos\theta =$ -1 $\tan\theta =$ 0 $\csc\theta =$ undef $\sec\theta =$ -1 $\cot\theta =$ undef	d. $\theta = \frac{3\pi}{2}$ (0, -1) $\sin\theta =$ -1 $\cos\theta =$ 0 $\tan\theta =$ undef $\csc\theta =$ -1 $\sec\theta =$ undef $\cot\theta =$ 0

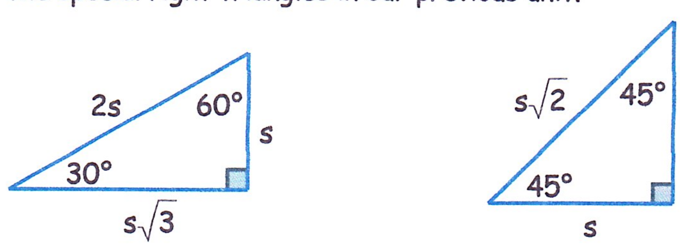
Now get out your "unit circle" and fill in the angle values and trig values for the quadrantal angles.

Example 6: State the quadrant(s) or axis in which θ terminates based on the given information.

a. $\sin\theta > 0$ S A T C Q I or II	b. $\sin\theta > 0, \cos\theta < 0$, S A T C Q II
c. $\csc\theta > 0$, $\tan\theta < 0$ $\sin\theta > 0$ S A T C Q II	d. $\tan\theta$ is undefined $\tan\theta = \frac{y}{x} \leftarrow = 0$ y-axis
e. $\sec\theta > 0$, $\cot\theta < 0$ $\cos\theta > 0$ $\tan\theta < 0$ S A T C Q IV	f. $\csc\theta$ is undefined $\leftarrow$ so $y = 0$ $\sin\theta = \frac{y}{r}$ x-axis
g. $\cos\theta = 0$ $\cos\theta = \frac{x}{r}$ so $x = 0$ y-axis	h. $\sin\theta = -1$ $\frac{y}{r}$ so $y = -1$ negative y-axis

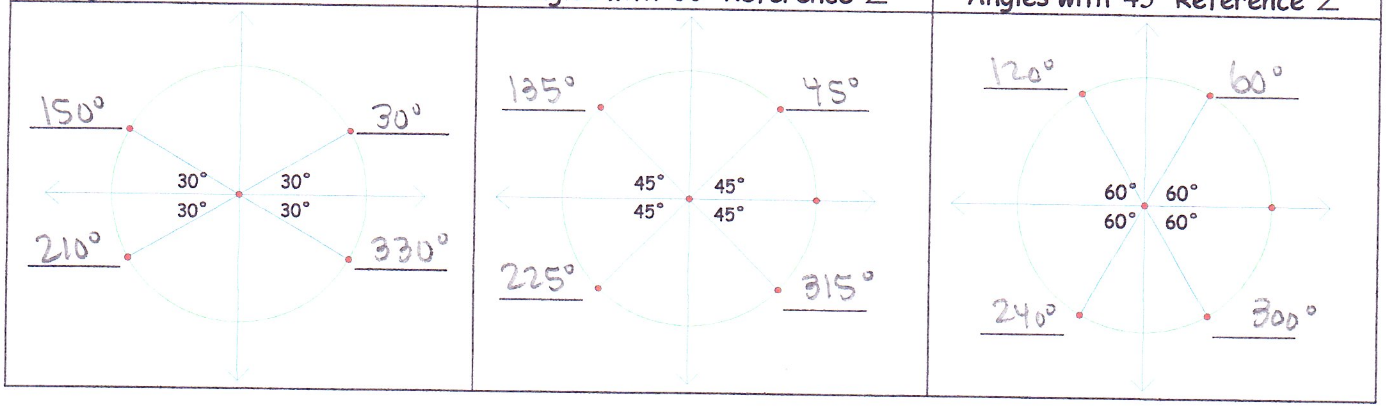
PART THREE:

Recall that we briefly saw the special right triangles in our previous unit.



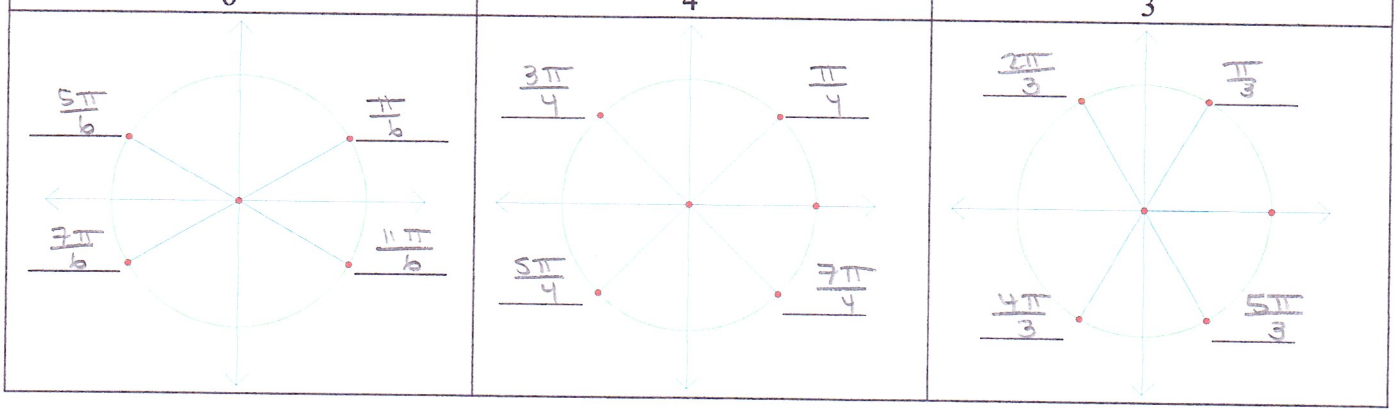
Let's find all of the angles measuring between 0° and 360° that have a reference angle of 30°, 45°, or 60°.

Angles with 30° Reference $\angle$	Angles with 60° Reference $\angle$	Angles with 45° Reference $\angle$



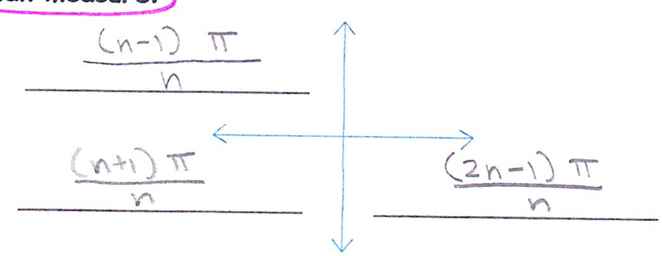
Now, with radians. Find all of the angles between 0 and 2π that have a reference angle of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$.

Angles with $\frac{\pi}{6}$ Reference $\angle$	Angles with $\frac{\pi}{4}$ Reference $\angle$	Angles with $\frac{\pi}{3}$ Reference $\angle$



Is there some kind of pattern to these angles in radian measure?

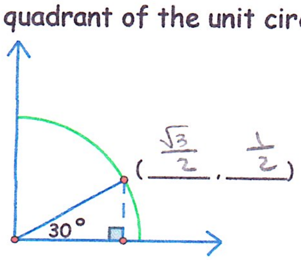
where n = 6, 4, or 3

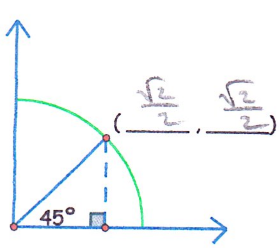


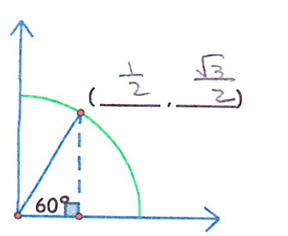
Now, let's put it all together on the front side of the **Unit Circle** you have been provided...

PART FOUR: More with the Unit Circle... and not the Quadrantal Angles! ☺

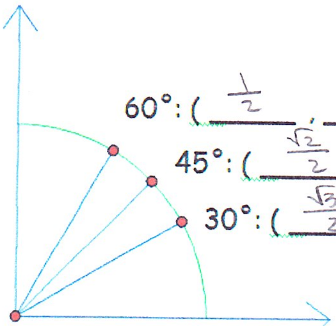
We were briefly introduced to the Unit Circle previously as we evaluated the trigonometric values for various quadrantal angles. But not every angle is a quadrantal angle, is it? Let's start by focusing just on the first quadrant of the unit circle.







So, let's put it all together to give us the first quadrant of the unit circle. ☺



60°: ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$)

45°: ($\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$)

30°: ($\frac{\sqrt{3}}{2}$, $\frac{1}{2}$)

MEMORIZE THIS NOW!

It will make your life easier. If you do not, then you will have a tough time of it for the rest of the semester.

Don't say I did not warn you! ☺

Let's now go back to our SEPARATE Unit Circle and fill in the remaining coordinates.

It will make your life TONS easier if you memorize the following pairs, which are reciprocals of each other.

$\frac{1}{2}$ and 2	$\frac{\sqrt{2}}{2}$ and $\sqrt{2}$	$\frac{\sqrt{3}}{2}$ and $\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$ and $\sqrt{3}$

Example 1: Using reference angles and the unit circle, find the values of the 6 trigonometric functions.

a. $\frac{\pi}{4}$ S | A T | C ref ∠ $\frac{\pi}{4}$ $\sin\theta = +\frac{\sqrt{2}}{2}$ $\cos\theta = +\frac{\sqrt{2}}{2}$ $\tan\theta = +1$ $\csc\theta = +\sqrt{2}$ $\sec\theta = +\sqrt{2}$ $\cot\theta = +1$	b. $\frac{5\pi}{3}$ S | A T | C ref ∠ $\frac{\pi}{3}$ $\sin\theta = -\frac{\sqrt{3}}{2}$ $\cos\theta = +\frac{1}{2}$ $\tan\theta = -\sqrt{3}$ $\csc\theta = -\frac{2\sqrt{3}}{3}$ $\sec\theta = +2$ $\cot\theta = -\frac{\sqrt{3}}{3}$
c. 150° S | A T | C ref ∠ 30° $\sin\theta = +\frac{1}{2}$ $\cos\theta = -\frac{\sqrt{3}}{2}$ $\tan\theta = -\frac{\sqrt{3}}{3}$ $\csc\theta = +2$ $\sec\theta = -\frac{2\sqrt{3}}{3}$ $\cot\theta = -\sqrt{3}$	d. $\frac{7\pi}{6}$ S | A T | C ref ∠ $\frac{\pi}{3}$ $\sin\theta = -\frac{1}{2}$ $\cos\theta = -\frac{\sqrt{3}}{2}$ $\tan\theta = +\frac{\sqrt{3}}{3}$ $\csc\theta = -2$ $\sec\theta = -\frac{2\sqrt{3}}{3}$ $\cot\theta = +\sqrt{3}$
e. −135° S | A T | C ref ∠ 45° $\sin\theta = -\frac{\sqrt{2}}{2}$ $\cos\theta = -\frac{\sqrt{2}}{2}$ $\tan\theta = +1$ $\csc\theta = -\sqrt{2}$ $\sec\theta = -\sqrt{2}$ $\cot\theta = +1$	f. −60° S | A T | C ref ∠ 60° $\sin\theta = -\frac{\sqrt{3}}{2}$ $\cos\theta = +\frac{1}{2}$ $\tan\theta = -\sqrt{3}$ $\csc\theta = -\frac{2\sqrt{3}}{3}$ $\sec\theta = +2$ $\cot\theta = -\frac{\sqrt{3}}{3}$

Example 2: The terminal side of an angle θ in standard position passes through $\left(-\frac{3}{5}, \frac{4}{5}\right)$ (x y). Determine the exact values of the 6 trigonometric functions of the angle θ.

$x^2 + y^2 = r^2$

$\frac{9}{25} + \frac{16}{25} = r^2$

$r^2 = 1 \longrightarrow r = 1$

on unit circle!

$\sin\theta = \frac{4}{5}$ $\cos\theta = -\frac{3}{5}$ $\tan\theta = -\frac{4}{3}$

$\csc\theta = \frac{5}{4}$ $\sec\theta = -\frac{5}{3}$ $\cot\theta = -\frac{3}{4}$

Finally, let's practice a bit more with evaluating trigonometric functions with our calculator. We did this type of problem in our previous unit, but it never hurts to revisit this skill again.

Example 3: Round each of the following to four decimal places. Watch your mode!

a. $\sin 73^\circ$ 0.9563	b. $\cos(-263^\circ)$ −0.1219	c. $\tan\frac{15\pi}{8}$ −0.4142	d. $\csc\frac{\pi}{12}$ $\frac{1}{\sin\frac{\pi}{12}}$ 3.8637
d. $\sec(-54^\circ)$ $\frac{1}{\cos(-54^\circ)}$ 1.7013	e. $\cot\left(-\frac{7\pi}{5}\right) = \frac{1}{\tan -\frac{7\pi}{5}}$ −0.3249	f. $\tan 301^\circ$ −1.6643	g. $\sec\frac{13\pi}{9} = \frac{1}{\cos\frac{13\pi}{9}}$ −5.7588

Remember... the calculator only does what you instruct it to do. If you tell it to find the wrong thing, it will.

PART FIVE: Domain and Period of Sine and Cosine

What is the DOMAIN of the sine and cosine functions? $\mathbb{R}$ or $(-\infty, \infty)$

What is the RANGE of the sine and cosine functions? $-1 \le y \le 1$ or $[-1, 1]$

Do all co-terminal angles have the same trig function values? yes

So, if we wanted to find the sine or cosine of an angle measuring 390° we would find sine or cosine of $390^\circ - 360^\circ$, or 30°.

If we wanted to find the sine or cosine of an angle measuring $\frac{11\pi}{3}$, we would find sine or cosine of $\frac{11\pi}{3} - 2\pi$, or $\frac{5\pi}{3}$.

As a matter of fact all angles of the form $\theta \pm 360^\circ n$ or $\theta \pm 2\pi n$ will have the same sine or cosine values as θ.
where n is an integer!

Hence, we say sine and cosine are **PERIODIC FUNCTIONS** and have a **PERIOD** of 360° or 2π radians.

Example 1: Evaluate the exact value of the given function at the given angle.

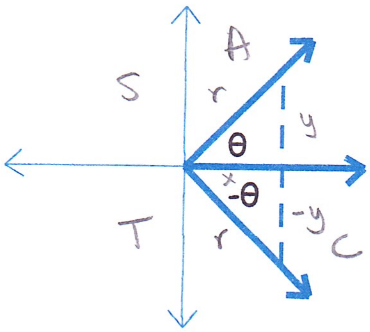
a. $\cos 420^\circ = \cos 60^\circ$ $= \boxed{\frac{\sqrt{3}}{2}}$	b. $\sin\left(-\frac{7\pi}{4}\right)$ $= \sin\frac{\pi}{4}$ $= \boxed{\frac{\sqrt{2}}{2}}$	c. $\cos\frac{8\pi}{3} = \cos\frac{2\pi}{3}$ $= \boxed{-\frac{1}{2}}$	d. $\sin\frac{21\pi}{4} = \sin\frac{5\pi}{4}$ $\boxed{-\frac{\sqrt{2}}{2}}$

Even and Odd Trigonometric Functions (Negative Angle Identities):

A function is **EVEN** if $f(-x) = f(x)$.

A function is **ODD** if $f(-x) = -f(x)$.

Which of the trig function(s) are odd? Let's look at θ and $-\theta$.



$\cos\theta = \underline{\frac{x}{r}}$ $\quad\cos(-\theta) = \frac{x}{r}$ same

$\sin\theta = \underline{\frac{y}{r}}$ $\quad\sin(-\theta) = -\frac{y}{r}$

$\tan\theta = \underline{\frac{y}{x}}$ $\quad\tan(-\theta) = -\frac{y}{x}$ opposite

Therefore, the odd trig. functions are <u>tan, cot, sin, csc</u> and the even trig. functions are <u>cos, sec</u>.

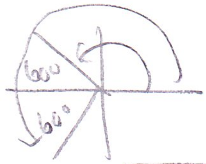
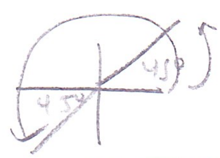
Example 2: Use the value of the given trigonometric function to evaluate the indicated function.

a. $\sin\theta = \frac{4}{5}$ ODD — opposite Find $\sin(-\theta)$ and $\csc(-\theta)$. $\sin(-\theta) = -\frac{4}{5}$, $\csc(-\theta) = -\frac{5}{4}$	b. $\cos\theta = \frac{2}{3}$ EVEN – same Find $\cos(-\theta)$ and $\sec(-\theta)$. $\cos(-\theta) = \frac{2}{3}$, $\sec(-\theta) = \frac{3}{2}$

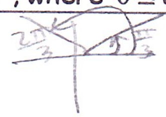
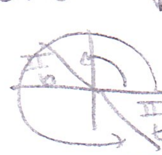
PART SIX: Given a function value, can you give the angle(s) that have that function value?

Let's look at $\sin\theta = \frac{1}{2}$. S A T C ref ∠ 30°	How many angles have that sine value? <u>an infinite amount</u> OK, then how many between 0° and 360°? <u>just 2</u> What are they? <u>30° and 150°</u>

Example 1: Solve for θ, where $0^\circ \le \theta < 360^\circ$.

a. $\cos\theta = -\frac{1}{2}$ S A T C ref ∠ 60°  $\boxed{120^\circ \text{ and } 240^\circ}$	b. $\tan\theta = 1$ S A T C ref ∠ 45°  $\boxed{45^\circ \text{ and } 225^\circ}$

Example 2: Solve for θ, where $0 \le \theta < 2\pi$.

c. $\csc\theta = \frac{2\sqrt{3}}{3}$ $\sin\theta = \frac{\sqrt{3}}{2}$ S A T C ref ∠ $\frac{\pi}{3}$  $\boxed{\frac{\pi}{3} \text{ and } \frac{2\pi}{3}}$	d. $\cot\theta = -\sqrt{3}$ $\quad\tan\theta = -\frac{1}{\sqrt{3}}$ S A T C ref ∠ $\frac{\pi}{6}$  $\boxed{\frac{5\pi}{6} \text{ and } \frac{11\pi}{6}}$